

Factorization of non-linear supersymmetry in one-dimensional Quantum Mechanics. II: proofs of theorems on reducibility

A.V. Sokolov

*V.A. Fock Institute of Physics,
Sankt-Petersburg State University*
E-mail: avs_avs@rambler.ru

Abstract

In this paper, we continue to study factorization of supersymmetric (SUSY) transformations in one-dimensional Quantum Mechanics into chains of elementary Darboux transformations with nonsingular coefficients. We define the class of potentials that are invariant under the Darboux – Crum transformations and prove a number of lemmas and theorems substantiating the formulated formerly conjectures on reducibility of differential operators for spectral equivalence transformations. Analysis of the general case is performed with all the necessary proofs.

1. Introduction

In this work, we present rigorous analysis of factorization of non-linear supersymmetric (SUSY) Quantum Mechanics [1] – [8] into really irreducible SUSY algebra elements, which can be used for construction of any polynomial SUSY algebra [9] with the help of the chain (ladder) construction [10] – [13]. From the viewpoint of the Darboux – Crum (almost) isospectral transformations [14] – [17], we consider factorization of an intertwining operator into a product of differential operators of first or second order with nonsingular real coefficients such that all the intermediate Hamiltonians have nonsingular real potentials. The hypothesis on the existence of such a factorization has been formulated earlier in [18, 19].

In [20], it was conjectured that it is possible to dress (multiply) an intertwining operator by a polynomial of the Hamiltonian preserving the same pair of (almost) isospectral Hamiltonians and so that the resultant operator may be factorized in the ladder way into nonsingular real blocks of first order in derivatives. In this part of the work, conditions for realization of such a program are found. This part continues the study started in [21] and we prove here two assertions formulated in [21] on reducibility of (almost) isospectral transformations into a chain sequence of irreducible blocks of first or second order in derivatives:

- (1) the assertion on reducibility of a nonminimizable intertwining operator with real spectrum of the matrix \mathbf{S} , multiplied by an appropriate polynomial of the Hamiltonian, into (a product of) intertwining operators of first order (Theorem 2);

- (2) the assertion on reducibility of a nonminimizable intertwining operator, whose matrix \mathbf{S} may have not only real but also complex eigenvalues, into (a product of) intertwining operators of first order and irreducible second-order intertwining operators of the I, II and III type [20, 21] (Theorem 3).

In what follows, we use the class K of potentials $V(x)$ with the following properties:

- (1) $V(x)$ is a real-valued function from the class $C_{\mathbb{R}}^{\infty}$;
- (2) there exist numbers $R_0 > 0$ and $\varepsilon > 0$ (depending on $V(x)$) such that the inequality $V(x) \geq \varepsilon$ takes place for any $|x| \geq R_0$;
- (3) the functions

$$\left(\int_{\pm R_0}^x \sqrt{|V(x_1)|} dx_1 \right)^2 \left(\frac{|V'(x)|^2}{|V(x)|^3} + \frac{|V''(x)|}{|V(x)|^2} \right) \quad (1)$$

are bounded for $x \geq R_0$ and $x \leq -R_0$, respectively.

In addition, we discuss normalizability and nonnormalizability of functions at $+\infty$ and/or at $-\infty$, as well as formal associated functions, which are defined as follows.

A function $f(x)$ is called *normalizable at $+\infty$ (at $-\infty$)* if there exists a real number R_+ (R_-) such that

$$\int_{R_+}^{+\infty} |f(x)|^2 dx < +\infty \quad \left(\int_{-\infty}^{R_-} |f(x)|^2 dx < +\infty \right). \quad (2)$$

Otherwise, $f(x)$ is called *nonnormalizable at $+\infty$ (at $-\infty$)*.

A function $\psi_{n,i}(x)$ is called a *formal associated function of i th order* of the Hamiltonian h for a spectral value λ_n if

$$(h - \lambda_n)^{i+1} \psi_{n,i} \equiv 0, \quad (h - \lambda_n)^i \psi_{n,i} \not\equiv 0. \quad (3)$$

The term “formal” emphasizes that this function is not necessarily normalizable (not necessarily belongs to $L_2(\mathbb{R})$). In particular, an associated function $\psi_{n,0}(x)$ of zero order is a formal eigenfunction of h (not necessarily a normalizable solution of the homogeneous Schrödinger equation).

The paper is organized as follows. At first, we present a number of assertions which clarify basic properties of Hamiltonians with potentials from the class K . These assertions are devoted to (i) invariance of the class K under intertwining, (ii) asymptotics of formal associated functions, (iii) properties of a sequence of formal associated functions under intertwining, and (iv) spectral properties of intertwined Hamiltonians. Next we prove auxiliary lemmas on reducibility of operators that intertwine Hamiltonians with potentials from the class K . At last, the main assertions (Theorems 2 and 3) on reducibility of above-mentioned operators are stated.

2. Basic properties of Hamiltonians with potentials from the class K

Proofs of all the lemmas presented in this section except Lemma 6 and of Theorem 1 are contained in [22, 23].

2.1. Invariance of the potential class K under intertwining

The invariance of the potential class K under intertwining is a corollary of the following lemma.

Lemma 1. *Assume that the following conditions are satisfied:*

- (1) $h^+ = -\partial^2 + V_1(x)$, $V_1(x) \in K$;
- (2) $h^- = -\partial^2 + V_2(x)$, where the potential $V_2(x)$ is real-valued and belongs to $C_{\mathbb{R}}$;
- (3) $q_N^- h^+ = h^- q_N^-$, where q_N^- is a differential operator of N th order with coefficients belonging to $C_{\mathbb{R}}^2$;
- (4) each eigenvalue of the matrix \mathbf{S} for the operator q_N^- satisfies one of the following conditions: either $\lambda \leq 0$ or $\text{Im } \lambda \neq 0$.

Then:

- (1) $V_2(x) \in K$;
- (2) coefficients of q_N^- belong to $C_{\mathbb{R}}^\infty$;
- (3) $h^+ q_N^+ = q_N^+ h^-$, where $q_N^+ = (q_N^-)^t$, and, moreover, coefficients of q_N^+ belong to $C_{\mathbb{R}}^\infty$ as well.

2.2. Asymptotics of formal associated functions

The asymptotic behavior of formal associated functions of a Hamiltonian h with a potential from the class K is described by the following lemma.

Lemma 2. *Assume that the following conditions are satisfied:¹*

- (1) $h = -\partial^2 + V(x)$, $V(x) \in K$;
- (2) $\lambda \in \mathbb{C}$ and either $\lambda \leq 0$ or $\text{Im } \lambda \neq 0$;
- (3) the branches of the functions $\sqrt{V(x) - \lambda}$ and $\sqrt[4]{V(x) - \lambda}$ are uniquely defined for $|x| \geq R_0$ by the condition $|\arg[V(x) - \lambda]| < \pi$;
- (4) $\xi_{\uparrow\downarrow}(x) = \pm \int_{\pm R_0}^x \sqrt{|V(x_1)|} dx_1$, $\xi_{\uparrow\downarrow}(x; \lambda) = \pm \int_{\pm R_0}^x \sqrt{V(x_1) - \lambda} dx_1$, $\eta_{\uparrow\downarrow}(x) = \pm \int_{\pm R_0}^x dx_1 / \sqrt{|V(x_1)|}$.

Then there exist denumerable sequences:

$\varphi_{n,\uparrow\downarrow}(x)$ of formal associated functions of h for a spectral value λ that are normalizable at $\pm\infty$,

and

$\hat{\varphi}_{n,\uparrow\downarrow}(x)$ of formal associated functions of h for a spectral value λ that are nonnormalizable at $\pm\infty$,

such that:

¹In what follows, the index \uparrow (\downarrow) corresponds to upper (lower) signs in the right-hand sides.

(1)

$$h\varphi_{0,\uparrow\downarrow} = \lambda\varphi_{0,\uparrow\downarrow}, \quad (h - \lambda)\varphi_{n,\uparrow\downarrow} = \varphi_{n-1,\uparrow\downarrow}, \quad n = 1, 2, 3, \dots, \quad (4)$$

$$h\hat{\varphi}_{0,\uparrow\downarrow} = \lambda\hat{\varphi}_{0,\uparrow\downarrow}, \quad (h - \lambda)\hat{\varphi}_{n,\uparrow\downarrow} = \hat{\varphi}_{n-1,\uparrow\downarrow}, \quad n = 1, 2, 3, \dots; \quad (5)$$

(2) if $\pm \int_{\pm R_0}^{\pm\infty} dx_1/\sqrt{|V(x_1)|} < +\infty$, then

$$\varphi_{n,\uparrow\downarrow}(x) = \frac{1}{n! \sqrt[4]{V(x) - \lambda}} \left(\pm \frac{1}{2} \int_{\pm\infty}^x \frac{dx_1}{\sqrt{V(x_1) - \lambda}} \right)^n e^{-\xi_{\uparrow\downarrow}(x;\lambda)} \left[1 + O\left(\frac{1}{\xi_{\uparrow\downarrow}(x)}\right) \right], \quad (6)$$

$$\hat{\varphi}_{n,\uparrow\downarrow}(x) = \frac{1}{n! \sqrt[4]{V(x) - \lambda}} \left(\mp \frac{1}{2} \int_{\pm\infty}^x \frac{dx_1}{\sqrt{V(x_1) - \lambda}} \right)^n e^{\xi_{\uparrow\downarrow}(x;\lambda)} \left[1 + O\left(\frac{1}{\xi_{\uparrow\downarrow}(x)}\right) \right], \quad (7)$$

$$\varphi'_{n,\uparrow\downarrow}(x) = \mp \frac{1}{n!} \sqrt[4]{V(x) - \lambda} \left(\pm \frac{1}{2} \int_{\pm\infty}^x \frac{dx_1}{\sqrt{V(x_1) - \lambda}} \right)^n e^{-\xi_{\uparrow\downarrow}(x;\lambda)} \left[1 + O\left(\frac{1}{\xi_{\uparrow\downarrow}(x)}\right) \right] \quad (8)$$

as $x \rightarrow \pm\infty$, $n = 0, 1, 2, \dots$;

(3) if $\pm \int_{\pm R_0}^{\pm\infty} dx_1/\sqrt{|V(x_1)|} = +\infty$, then

$$\varphi_{n,\uparrow\downarrow}(x) = \frac{1}{n! \sqrt[4]{V(x) - \lambda}} \left(\pm \frac{1}{2} \int_{\pm R_0}^x \frac{dx_1}{\sqrt{V(x_1) - \lambda}} \right)^n e^{-\xi_{\uparrow\downarrow}(x;\lambda)} \left[1 + O\left(\frac{\ln \eta_{\uparrow\downarrow}(x)}{\eta_{\uparrow\downarrow}(x)}\right) \right], \quad (9)$$

$$\hat{\varphi}_{n,\uparrow\downarrow}(x) = \frac{1}{n! \sqrt[4]{V(x) - \lambda}} \left(\mp \frac{1}{2} \int_{\pm R_0}^x \frac{dx_1}{\sqrt{V(x_1) - \lambda}} \right)^n e^{\xi_{\uparrow\downarrow}(x;\lambda)} \left[1 + O\left(\frac{\ln \eta_{\uparrow\downarrow}(x)}{\eta_{\uparrow\downarrow}(x)}\right) \right], \quad (10)$$

$$\varphi'_{n,\uparrow\downarrow}(x) = \mp \frac{1}{n!} \sqrt[4]{V(x) - \lambda} \left(\pm \frac{1}{2} \int_{\pm R_0}^x \frac{dx_1}{\sqrt{V(x_1) - \lambda}} \right)^n e^{-\xi_{\uparrow\downarrow}(x;\lambda)} \left[1 + O\left(\frac{\ln \eta_{\uparrow\downarrow}(x)}{\eta_{\uparrow\downarrow}(x)}\right) \right] \quad (11)$$

as $x \rightarrow \pm\infty$, $n = 0, 1, 2, \dots$.

Corollary 1. If $V(x) \in K$ and $\text{Im } \lambda \neq 0$, then, in view of (6) and (8) ((9) and (11)),²

$$\lim_{x \rightarrow \pm\infty} [\varphi'_{0,\uparrow\downarrow}(x)\varphi_{0,\uparrow\downarrow}^*(x) - \varphi_{0,\uparrow\downarrow}(x)\varphi_{0,\uparrow\downarrow}'^*(x)] =$$

$$\lim_{x \rightarrow \pm\infty} \left\{ e^{-2\text{Re } \xi_{\uparrow\downarrow}(x;\lambda)} \left[\mp \frac{\sqrt[4]{V(x) - \lambda}}{\sqrt[4]{V(x) - \lambda^*}} (1 + o(1)) \pm \frac{\sqrt[4]{V(x) - \lambda^*}}{\sqrt[4]{V(x) - \lambda}} (1 + o(1)) \right] \right\} = 0. \quad (12)$$

Thus, if $V(x) \in K$ and $\text{Im } \lambda \neq 0$, then the Wronskian W of a function (denoted below φ) from $\ker(h - \lambda)$ that is normalizable at one of infinities and of the complex conjugate function tends to zero at the same infinity; in addition, due to the monotonicity of iW ($iW' = i(\varphi''\varphi^* - \varphi\varphi''^*) = 2\text{Im } \lambda|\varphi|^2$), the Wronskian does not have zeroes. Let us note

²To prove rigorously that the limit in (12) is equal to zero, it is sufficient to use, in addition to (12), the second point of the definition of K and the estimate $\text{Re } \sqrt{V(x) - \lambda} \geq C_\lambda \sqrt{|V(x)|}$ derived in [23]. This estimate is correct under fixed $V(x) \in K$ and λ (with $\lambda \geq 0$ or $\text{Im } \lambda \neq 0$) for some $C_\lambda > 0$ and any $|x| \geq R_0$.

that, in the general case (contrary to [24]), this statement is not always valid. For example, the Hamiltonian

$$h = -\partial^2 - \alpha^2 e^{2\beta x} + 2\alpha\delta e^{\beta x}, \quad \alpha \in \mathbb{R}, \quad \beta > 0, \quad \alpha\delta > 0,$$

has a formal eigenfunction for the spectral value $\lambda = \delta^2 - \frac{\beta^2}{4} - i\beta\delta$ of the form $\varphi(x) = \exp[i\frac{\alpha}{\beta}e^{\beta x} - (i\delta + \frac{\beta}{2})x]$. This function tends exponentially to zero as $x \rightarrow +\infty$, but at the same time, the Wronskian

$$W(x) = \varphi'(x)\varphi^*(x) - \varphi(x)\varphi'^*(x) = 2i(\alpha - \delta e^{-\beta x})$$

does not tend to zero as $x \rightarrow +\infty$ and has a real root.

Corollary 2. Under the conditions of Lemma 2, any formal associated function of h of n th order normalizable at $\pm\infty$, for a spectral value λ such that either $\lambda \leq 0$ or $\text{Im } \lambda \neq 0$, can be written in the form

$$\sum_{j=0}^n a_{j,\uparrow\downarrow} \varphi_{j,\uparrow\downarrow}(x), \quad a_{j,\uparrow\downarrow} = \text{Const}, \quad a_{n,\uparrow\downarrow} \neq 0, \quad (13)$$

and any associated function of h of n th order, nonnormalizable at $\pm\infty$, for the same spectral value λ can be presented as follows:

$$\sum_{j=0}^n (b_{j,\uparrow\downarrow} \varphi_{j,\uparrow\downarrow}(x) + c_{j,\uparrow\downarrow} \hat{\varphi}_{j,\uparrow\downarrow}(x)), \quad (14)$$

where $b_{j,\uparrow\downarrow}, c_{j,\uparrow\downarrow} = \text{Const}$ and either $b_{n,\uparrow\downarrow} \neq 0$ or $c_{n,\uparrow\downarrow} \neq 0$.

2.3. Action of an intertwining operator on a sequence of formal associated functions

Properties of a sequence of formal associated functions under intertwining are described by the following lemma.

Lemma 3. *Assume that:*

- (1) *the conditions of Lemma 1 are satisfied;*
- (2) *$\varphi_n(x)$, $n = 0, \dots, M$, is a sequence of formal associated functions of h^+ for a spectral value λ :*

$$h^+ \varphi_0 = \lambda \varphi_0, \quad (h^+ - \lambda) \varphi_n = \varphi_{n-1}, \quad n = 1, \dots, M,$$

where either $\lambda \leq 0$ or $\text{Im } \lambda \neq 0$.

Then:

- (1) *there is a number m , $0 \leq m \leq \min\{M+1, N\}$, such that*

$$q_N^- \varphi_n \equiv 0, \quad n = 0, \dots, m-1,$$

and

$$\psi_l = q_N^- \varphi_{m+l}, \quad l = 0, \dots, M-m,$$

is a sequence of formal associated functions of h^- for the spectral value λ :

$$h^- \psi_0 = \lambda \psi_0, \quad (h^- - \lambda) \psi_l = \psi_{l-1}, \quad l = 1, \dots, M-m;$$

- (2) if a function $\varphi_n(x)$, for a given $0 \leq n \leq M$, is normalizable at $+\infty$ (at $-\infty$), then $q_N^- \varphi_n$ is normalizable at $+\infty$ (at $-\infty$) as well.

Corollary 3. The Hamiltonian h^+ is an intertwining operator for itself, and both eigenvalues of its matrix \mathbf{S} are zero. Hence, if $\varphi_n(x)$ is normalizable at $+\infty$ (at $-\infty$), then the functions $\varphi_j(x)$, $j = 0, \dots, n-1$, are normalizable at $+\infty$ (at $-\infty$) as well.

Corollary 4. Assume that $\varphi_{i,j}^-(x)$ is a canonical basis in $\ker q_N^-$, i.e., the matrix \mathbf{S} of the operator q_N^- has in this basis the canonical (Jordan) form:

$$h^+ \varphi_{i,0}^- = \lambda_i \varphi_{i,0}^-, \quad (h^+ - \lambda_i) \varphi_{i,j}^- = \varphi_{i,j-1}^-, \quad i = 1, \dots, n, \quad j = 1, \dots, k_i - 1, \quad \sum_{i=1}^n k_i = N.$$

Then there are numbers $k_{i\uparrow}^+$ and $k_{i\downarrow}^+$ such that $0 \leq k_{i\uparrow,\downarrow}^+ \leq k_i$ and for any i , the functions

$$\varphi_{i,j}^-(x), \quad j = 0, \dots, k_{i\uparrow,\downarrow}^+ - 1,$$

are normalizable at $\pm\infty$, and the functions

$$\varphi_{i,j}^-(x), \quad j = k_{i\uparrow,\downarrow}^+, \dots, k_i - 1,$$

are nonnormalizable at the same $\pm\infty$. Independence of these numbers $k_{i\uparrow,\downarrow}^+$ on a choice of the canonical basis in the case, where the intertwining operator q_N^- is nonminimizable, is a corollary of the following lemma.

Lemma 4. Assume that:

- (1) the conditions of Lemma 1 are satisfied;
- (2) q_N^- is nonminimizable.

Then any two formal associated functions of h^+ of the same order for the same spectral value λ when being elements of $\ker q_N^-$ are either simultaneously normalizable at $+\infty$ or simultaneously nonnormalizable at $+\infty$. The same fact takes place at $-\infty$.

2.4. Statements on spectra of intertwined Hamiltonians

The following Lemma 5 clarifies an interrelation between the behavior at $\pm\infty$ of elements of canonical bases for mutually transposed intertwining operators.

Lemma 5. Assume that:

- (1) the conditions of Lemma 1 are satisfied;
- (2) q_N^- is nonminimizable;
- (3) k_i is algebraic multiplicity of an eigenvalue λ_i of the matrix \mathbf{S} of the operator q_N^- ;
- (4) $\{\varphi_{i,j}^-\}$ and $\{\varphi_{i,j}^+\}$ are canonical bases of $\ker q_N^-$ and $\ker q_N^+$, respectively:

$$h^\pm \varphi_{i,0}^\mp = \lambda_i \varphi_{i,0}^\mp, \quad (h^\pm - \lambda_i) \varphi_{i,j}^\mp = \varphi_{i,j-1}^\mp, \quad i = 1, \dots, n, \quad j = 1, \dots, k_i - 1, \quad \sum_{i=1}^n k_i = N.$$

Then for any i and j , the function $\varphi_{i,j}^-(x)$ is normalizable (nonnormalizable) at $+\infty$ if and only if $\varphi_{i,k_i-j-1}^+(x)$ is nonnormalizable (normalizable) at $+\infty$. The same fact takes place at $-\infty$.

Corollary 5. In the family $\varphi_{i,j}^-(x)$ ($\varphi_{i,j}^+(x)$) (with a fixed i) only the function $\varphi_{i,k_i-1}^-(x)$ ($\varphi_{i,k_i-1}^+(x)$) may be nonnormalizable at both infinities. Thus, in view of Corollary 3, one of the numbers $k_{i\uparrow,\downarrow}^+$ (with a fixed i) is not less than $k_i - 1$ and the other one is not greater than 1. Accordingly, the functions $\varphi_{i,j}^-(x)$ ($\varphi_{i,j}^+(x)$), $j = 1, \dots, k_i - 2$, are normalizable at one of infinities (the same for any j) and are nonnormalizable at the other infinity. Moreover, if the functions $\varphi_{i,j}^-(x)$ are normalizable at $+\infty$ (at $-\infty$), then the functions $\varphi_{i,j}^+(x)$ are normalizable at $-\infty$ (at $+\infty$).

Corollary 6. If $\varphi_{i,0}^-(x)$ ($\varphi_{i,0}^+(x)$) is nonnormalizable at both infinities, then $k_i = 1$.

Corollary 7. If both functions $\varphi_{i,0}^-(x)$ and $\varphi_{i,0}^+(x)$ are normalizable at both infinities, then $k_i \geq 2$.

Corollary 8. If $\text{Im } \lambda_i \neq 0$, then, in view of Corollary 3, the functions $\varphi_{i,j}^-(x)$ ($\varphi_{i,j}^+(x)$), $j = 0, \dots, k_i - 1$, are normalizable at one of infinities (the same for any j) and nonnormalizable at the other infinity. Moreover, if the functions $\varphi_{i,j}^-(x)$ are normalizable at $+\infty$ (at $-\infty$), then functions $\varphi_{i,j}^+(x)$ are normalizable at $-\infty$ (at $+\infty$).

The following theorem indicates an interrelation between spectra of intertwined Hamiltonians and the behavior at $\pm\infty$ of elements of a canonical basis in the intertwining operator kernel.

Theorem 1 (Index Theorem). *Assume that the conditions of Lemma 5 are satisfied. Set $\nu_{\pm}(\lambda) = 1$ if λ is an eigenvalue of h^{\pm} and $\nu_{\pm}(\lambda) = 0$ otherwise. As well set $n_{\pm}(\lambda) = n_0(\lambda) = 0$ if λ is not an eigenvalue of the matrix \mathbf{S} of the operator q_N^- . If $\lambda = \lambda_i$ (where λ_i is an eigenvalue of the matrix \mathbf{S} of the operator q_N^-), let $n_{\pm}(\lambda_i)$ be the number of functions from the family $\varphi_{i,j}^{\mp}(x)$, $j = 0, \dots, k_i - 1$, that are normalizable at both infinities and let $n_0(\lambda_i)$ be the number of functions from the family $\varphi_{i,j}^{\mp}(x)$, $j = 0, \dots, k_i - 1$, that are normalizable only at one of infinities. Then for any λ such that either $\lambda \leq 0$ or $\text{Im } \lambda \neq 0$ the equality*

$$\nu_+(\lambda) - n_+(\lambda) = \nu_-(\lambda) - n_-(\lambda) \quad (15)$$

takes place. Moreover, if $n_0(\lambda) > 0$ for some λ , then

$$\nu_+(\lambda) - n_+(\lambda) = \nu_-(\lambda) - n_-(\lambda) = 0$$

for this λ .

The next lemma indicates an interrelation between the inclusion of a bound state wave function of a Hamiltonian to the kernel of an intertwining operator and the inclusion of the energy of the same state to the spectrum of the matrix \mathbf{S} of the considered intertwining operator.

Lemma 6. *If the conditions of Lemma 1 are satisfied, then a wave function of a bound state of h^{\pm} belongs to $\ker q_N^{\mp}$ if and only if the energy of this bound state is contained in the spectrum of the matrix \mathbf{S} of the operator q_N^{\mp} .*

Proof. We only consider the case of h^+ and q_N^- since the case of h^- and q_N^+ is treated similarly.

NECESSITY. Assume that h^+ has a bound state with energy E which is described by a wave function $\varphi(x)$ and that, in addition, $q_N^-\varphi = 0$. We claim that E belongs to the spectrum of the matrix \mathbf{S} of the operator q_N^- . Let λ_i be an eigenvalue of the matrix \mathbf{S} of

the operator q_N^- of algebraic multiplicity k_i , $i = 1, \dots, n$, so that $k_1 + \dots + k_n = N$. By Theorem 1 of [21],

$$0 = q_N^+ q_N^- \varphi = \prod_{i=1}^n (h^+ - \lambda_i)^{k_i} \varphi = \prod_{i=1}^n (E - \lambda_i)^{k_i} \varphi, \quad (16)$$

from which it follows that E belongs to the spectrum of the matrix \mathbf{S} of the operator q_N^- .

SUFFICIENCY. We assume now that E belongs to the spectrum of the matrix \mathbf{S} of the operator q_N^- . Let us show that $q_N^- \varphi = 0$. Let

$$q_N^- = p_M^- P(h^+), \quad (17)$$

where $P(h^+)$ is a polynomial and p_M^- is a nonminimizable operator which intertwines h^+ and h^- ($p_M^- h^+ = h^- p_M^-$). If E is a zero of P , then the statement is proved. Let us proceed to the case $P(E) \neq 0$. In this case, E belongs to the spectrum of the matrix \mathbf{S} of the operator p_M^- , because by Theorem 1 of [21], the spectrum of the matrix \mathbf{S} of the operator q_N^- coincides with the set of zeroes of the polynomial $P_N(h^+) = q_N^+ q_N^-$, the spectrum of the matrix \mathbf{S} of the operator p_M^- coincides with the set of zeroes of the polynomial $P_M(h^+) = p_M^+ p_M^-$, $p_M^+ = (p_M^-)^t$ and

$$P_N(h^+) = q_N^+ q_N^- = P(h^+) p_M^+ p_M^- P(h^+) = P^2(h^+) P_M(h^+). \quad (18)$$

Let $q_N^- \varphi \neq 0$. Then $p_M^- \varphi \neq 0$ as well, since otherwise $q_N^- \varphi = P(h^-) p_M^- \varphi = 0$. By Lemma 3, $p_M^- \varphi$ is an eigenfunction of h^- that belongs to $\ker p_M^+$ since $p_M^+ p_M^- \varphi = P_M(E) \varphi = 0$. The latter fact contradicts statement (15) of Theorem 1, because in the case under consideration, $\nu_+(E) = \nu_-(E) = 1$, and by Lemmas 4 and 5, $n_+(E) = 0$ and $n_-(E) = 1$ (in the considered case, $n_{\pm}(E)$ corresponds to p_M^- and not to q_N^-). Lemma 6 is proved.

Corollary 9. By Lemmas 5 and 6, h^{\pm} has a bound state at a level $E = \lambda_i$ if and only if the function $\varphi_{i,k_i-1}^+(x)$ ($\varphi_{i,k_i-1}^-(x)$) is nonnormalizable at both infinities.

Corollary 10. Assume that at least one of the coefficients of q_N^{\mp} has a nontrivial imaginary part and that k_N^{\mp} and p_M^{\mp} are differential operators with real-valued coefficients such that $q_N^{\mp} = k_N^{\mp} + i p_M^{\mp}$. Then (see [25]) the operators k_N^{\mp} and p_M^{\mp} intertwine the same Hamiltonians as q_N^{\mp} . Moreover, since a wave function of a bound state can be chosen real-valued, any wave function of a bound state that belongs to $\ker q_N^{\mp}$ belongs to $\ker k_N^{\mp}$ and $\ker p_M^{\mp}$ as well. Hence, any eigenvalue of the matrix \mathbf{S} of the operator q_N^{\mp} , which is the energy of a bound state of h^{\pm} , belongs to the spectra of the matrices \mathbf{S} of the operators k_N^{\mp} and p_M^{\mp} as well.

3. Lemmas on partial reducibility of intertwining operators

Lemma 7. Assume that:

- (1) the conditions of Lemma 5 are satisfied;
- (2) all coefficients of q_N^- are real-valued;
- (3) $\text{Im } \lambda_l \neq 0$.

Then q_N^- can be represented as the product of two intertwining operators k_{N-2}^- and p_2^- , so that:

$$(1) \quad q_N^- = k_{N-2}^- p_2^-, \quad h_0 p_2^- = p_2^- h^+, \quad h^- k_{N-2}^- = k_{N-2}^- h_0, \quad (19)$$

where h_0 is the Hamiltonian with the potential from K ;

(2) p_2^- is the really irreducible intertwining operator of second order of the I type with real-valued coefficients from $C_{\mathbb{R}}^\infty$, and the spectrum of the matrix \mathbf{S} of the operator p_2^- consists of λ_l and λ_l^* ;

(3) k_{N-2}^- is the intertwining operator of $(N-2)$ th order with real-valued coefficients from $C_{\mathbb{R}}^\infty$.

Proof. Taking into account reality of coefficients of q_N^- , we assume, without loss of generality, that a basis $\{\varphi_{i,j}^-\}$ in the kernel of q_N^- is chosen so that the functions $\varphi_{i,j}^-$, corresponding to real λ_i , are real-valued and the functions $\varphi_{i,j}^-$ and $\varphi_{k,j}^-$, corresponding to complex conjugated numbers λ_i and $\lambda_k = \lambda_i^*$, are related by $\varphi_{k,j}^- = \varphi_{i,j}^{-*}$.

Using the procedure described in Lemma 1 of [25], one can represent q_N^- in the form

$$q_N^- = k_{N-2}^- p_2^-, \quad (20)$$

where p_2^- is the differential operator of second order whose kernel basis consists of $\varphi_{l,0}^-(x)$ and $\varphi_{l,0}^{-*}(x)$, and k_{N-2}^- is the differential operator of $(N-2)$ th order whose kernel basis consists of $p_2^- \varphi_{i,j}^-$, with the exception of $p_2^- \varphi_{l,0}^-$ and $p_2^- \varphi_{l,0}^{-*}$. Moreover, by the above-mentioned lemma, p_2^- and k_{N-2}^- intertwine h^+ and h^- , respectively, with certain Hamiltonian h_0 , so that equalities (19) hold.

Let us denote the Wronskians of elements of the above-mentioned bases in $\ker k_{N-2}^-$ and $\ker p_2^-$ by $W_k(x)$ and $W_p(x)$, respectively. Then, by formula (11) of [21], the potential $V_0(x)$ of the Hamiltonian h_0 is related to $V_{1,2}(x)$ by the following equalities:

$$V_0(x) = V_1(x) - 2[\ln W_p(x)]'', \quad V_2(x) = V_0(x) - 2[\ln W_k(x)]''. \quad (21)$$

In view of Corollary 8 the function $\varphi_{l,0}^-(x)$ is normalizable at one of infinities. Thus, by Corollary 1, the Wronskian $W_p(x)$ does not have zeroes. We derive from this fact and from the inclusion $\varphi_{l,0}^-(x) \in C_{\mathbb{R}}^\infty$ that $V_0(x)$ (see (21)) and coefficients of

$$p_2^- = \frac{1}{W_p(x)} \begin{vmatrix} \varphi_{l,0}^{-*}(x) & \varphi_{l,0}^{-*'}(x) & \varphi_{l,0}^{-*''}(x) \\ \varphi_{l,0}^-(x) & \varphi_{l,0}^{-'}(x) & \varphi_{l,0}^{-''}(x) \\ 1 & \partial & \partial^2 \end{vmatrix} \quad (22)$$

belong to $C_{\mathbb{R}}^\infty$. Moreover, coefficients of p_2^- are real since complex conjugation of these coefficients is equivalent to permutations of two lines in both determinants in (22). The fact that $V_0(x)$ is real-valued follows from (21) and from the fact that $W_p(x)$ is evidently purely imaginary. The inclusion of $V_0(x)$ into K follows from the statements proved above and from Lemma 1.

The absence of zeroes for $W_k(x)$ follows from the infinite smoothness of $V_0(x)$ and $V_2(x)$ and from the fact that the general solution of (21) has the form

$$W_k(x) = C_1 \exp \left\{ C_2 x + \frac{1}{2} \int_0^x dx_1 \int_0^{x_1} dx_2 [V_0(x_2) - V_2(x_2)] \right\},$$

where $C_1 \neq 0$ and C_2 are constants. The infinite smoothness of coefficients of k_{N-2}^- follows from the absence of zeroes for $W_k(x)$, from the infinite smoothness of $W_k(x)$ (the Wronskian of functions from $C_{\mathbb{R}}^\infty$), and from the formula for k_{N-2}^- similar to (22). The fact that coefficients of k_{N-2}^- are real-valued is an obvious corollary of the fact that coefficients of q_N^- and p_2^- are real-valued. Lemma 7 is proved.

Lemma 8. *Assume that:*

- (1) *the conditions of Lemma 5 are satisfied;*
- (2) *λ_M is the least real eigenvalue of the matrix \mathbf{S} of the operator q_N^- ;*
- (3) *λ_M is situated below the energy of the ground state of h^- .*

Then q_N^- can be factorized into the product of two intertwining operators k_{N-1}^- and p_1^- , so that:

$$(1) \quad q_N^- = k_{N-1}^- p_1^-, \quad h_0 p_1^- = p_1^- h^+, \quad h^- k_{N-1}^- = k_{N-1}^- h_0, \quad (23)$$

where h_0 is the Hamiltonian with the potential from K ;

- (2) *p_1^- is the intertwining operator of first order with real-valued coefficients from $C_{\mathbb{R}}^\infty$, and its matrix \mathbf{S} consists of λ_M ;*
- (3) *k_{N-1}^- is the intertwining operator of $(N-1)$ th order with coefficients from $C_{\mathbb{R}}^\infty$;*
- (4) *if λ_M is (is not) the energy of a bound state of h^+ , then an element of a basis in $\ker p_1^-$ is normalizable at both infinities (at one of infinities only).*

If coefficients of q_N^- are real-valued, then coefficients of k_{N-1}^- are real-valued as well.

Proof. By the conditions of our lemma and by Lemmas 3 and 6, the number λ_M is not situated above the energy of the ground state of h^+ . By our conditions and by Lemma 5, the function $\varphi_{M,0}^-(x)$ cannot be nonnormalizable at both infinities. At the same time, consider a formal eigenfunction of h^+ that is normalizable at least at one of infinities and corresponds to a spectral value that is not situated above the energy of the ground state. Such a function has no zeroes and may differ from a real-valued function (if such a difference exists) by a constant factor only. Hence, the operator

$$p_1^- = \partial - \frac{\varphi_{M,0}^{-'}}{\varphi_{M,0}^-} \quad (24)$$

has real-valued coefficients from $C_{\mathbb{R}}^\infty$. In accordance with the procedure described in Lemma 1 of [25], this operator can be separated from q_N^- , so that the equalities (23) are valid, where k_{N-1}^- is the intertwining operator of $(N-1)$ th order with coefficients from $C_{\mathbb{R}}^\infty$, and h_0 is the Hamiltonian whose potential $V_0(x)$, by the relation (11) of [21], is equal to

$$V_0(x) = V_1(x) - 2[\ln \varphi_{M,0}^-(x)]''. \quad (25)$$

We deduce that $V_0(x)$ is real-valued and infinitely smooth from relation (25), from the absence of zeroes for $\varphi_{M,0}^-(x)$, from the inclusions $V_1(x) \in K$ and $\varphi_{M,0}^-(x) \in C_{\mathbb{R}}^\infty$, and from the proportionality of $\varphi_{M,0}^-(x)$ to a real-valued function. The inclusion of $V_0(x)$ into the class K follows from the statements proven above and from Lemma 1. The fourth

statement of the lemma follows from the normalizability of $\varphi_{M,0}^-(x)$ at least at one of infinities and from Lemma 6. If coefficients of q_N^- are real-valued, then coefficients of k_{N-1}^- are obviously real-valued as well. Lemma 8 is proved.

Lemma 9. *Assume that:*

- (1) *the potential of the Hamiltonian h^+ belongs to K ; the potential of the Hamiltonian h_1 is real-valued and belongs to $C_{\mathbb{R}}^1$; the potential of the Hamiltonian h^- is real-valued and belongs to $C_{\mathbb{R}}$;*
- (2) *$\varphi_0(x)$ is a wave function of the ground state of h^+ , so that*

$$h^+ \varphi_0 = E_{0+} \varphi_0, \quad E_{0+} \leq 0; \quad (26)$$

the Hamiltonians h^+ and h_1 are intertwined by the operator $p_{11}^- = \partial - \varphi_0'/\varphi_0$, so that

$$p_{11}^- h^+ = h_1 p_{11}^-; \quad (27)$$

- (3) *$\psi(x)$ is a function that is normalizable at one of infinities only and belongs to $\ker(h_1 - \lambda)$, $\lambda < E_{0+}$; the Hamiltonians h_1 and h^- are intertwined by the operator $k_{11}^- = \partial - \psi'/\psi$, so that*

$$k_{11}^- h_1 = h^- k_{11}^-. \quad (28)$$

Then:

- (1) *the potentials of h^- and h_1 belong to K ; coefficients of p_{11}^- and k_{11}^- are real-valued and belong to $C_{\mathbb{R}}^\infty$;*
- (2) *the function $(p_{11}^-)^t \psi$ does not have zeroes, belongs to $\ker(h^+ - \lambda)$, and is normalizable at one of infinities only (the same as ψ);*
- (3) *the operator*

$$p_{12}^- = \partial - \frac{(p_{11}^+ \psi)'}{p_{11}^+ \psi}, \quad p_{11}^+ = (p_{11}^-)^t \quad (29)$$

has real-valued coefficients from $C_{\mathbb{R}}^\infty$ and intertwines h^+ with the Hamiltonian $h_2 = \lambda + p_{12}^-(q_{12}^-)^t$, so that

$$p_{12}^- h^+ = h_2 p_{12}^-; \quad (30)$$

the potential of h_2 belongs to K ; the matrix \mathbf{S} of the operator p_{12}^- consists of λ ;

- (4) *$p_{12}^- \varphi_0$ is a wave function of the ground state of h_2 with the energy E_{0+} ;*
- (5) *the operator*

$$k_{12}^- = \partial - \frac{(p_{12}^- \varphi_0)'}{p_{12}^- \varphi_0} \quad (31)$$

has real-valued coefficients from $C_{\mathbb{R}}^\infty$ and intertwines h_2 with h^- , so that

$$k_{12}^- h_2 = h^- k_{12}^-; \quad (32)$$

the matrix \mathbf{S} of the operator k_{12}^- consists of E_{0+} ;

- (6) *the equality*

$$k_{11}^- p_{11}^- = k_{12}^- p_{12}^- \quad (33)$$

holds.

Lemma 9 follows trivially from Lemmas 1 and 3, Theorem 1 of [21], standard construction which describes intertwining of Hamiltonians by operators of first order [1] – [8], [10] – [13], [20], and elementary information on zeroes of formal eigenfunctions of a Hamiltonian [26].

Lemma 10. *Assume that:*

- (1) *the conditions of Lemma 1 are satisfied with $N = 3$;*
- (2) *q_3^- is nonminimizable, and its coefficients are real-valued;*
- (3) *λ is the least real eigenvalue of the matrix \mathbf{S} of the operator q_3^- .*

Then there exist intertwining operators p_1^\pm and k_1^\pm of first orders and p_2^\pm and k_2^\pm of second orders such that:

- (1) *p_1^\pm , k_1^\pm , p_2^\pm and k_2^\pm have real-valued coefficients from $C_\mathbb{R}^\infty$;*

- (2)
$$p_1^+ = (p_1^-)^t, \quad k_1^+ = (k_1^-)^t, \quad p_2^+ = (p_2^-)^t, \quad k_2^+ = (k_2^-)^t; \quad (34)$$

- (3) *the matrices \mathbf{S} of the operators p_1^\pm and k_1^\pm consist of λ ;*

- (4)
$$q_3^- = k_2^- p_1^- = k_1^- p_2^-, \quad q_3^+ = p_1^+ k_2^+ = p_2^+ k_1^+, \quad (35)$$

and the potentials of intermediate Hamiltonians that correspond to these factorizations belong to K ;

- (5) *if the algebraic multiplicity of λ in the spectrum of the matrix \mathbf{S} of the operator q_3^- is equal to one, then an element of a basis in $\ker p_1^-$ is normalizable (nonnormalizable) at $+\infty$ if and only if an element of a basis in $\ker k_1^-$ is normalizable (nonnormalizable) at $+\infty$; the same fact is true at $-\infty$; the same facts take place for p_1^+ and k_1^+ .*

Proof. The first four statements of the lemma follow from Theorem 3 of [21] and from Lemma 1. In the proof of the fifth statement of the lemma, we consider the case of p_1^- , k_1^- and $x \rightarrow +\infty$ only, since the remaining cases can be examined analogously. Let $\varphi(x)$ be an element of a basis in $\ker p_1^-$. As the matrix \mathbf{S} of the operator p_1^- consists of λ , so $h^+ \varphi = \lambda \varphi$. Let also λ_1 and λ_2 be the remaining two eigenvalues of the matrix \mathbf{S} of the operator q_3^- as well as $\varphi_1(x)$ and $\varphi_2(x)$ be the remaining two elements of a canonical basis in $\ker q_3^-$, where, by condition, $\lambda \neq \lambda_{1,2}$. As $q_3^- = k_1^- p_2^-$, $q_3^- \varphi = 0$, and the basis in $\ker p_2^-$ consists of $\varphi_1(x)$ and $\varphi_2(x)$, so

$$\psi = p_2^- \varphi \neq 0 \quad (36)$$

is the only element of the basis in $\ker k_1^-$. On the other hand, by Theorem 1 of [21],

$$p_2^+ \psi = p_2^+ p_2^- \varphi = (h^+ - \lambda_1)(h^+ - \lambda_2) \varphi = (\lambda - \lambda_1)(\lambda - \lambda_2) \varphi \neq 0. \quad (37)$$

It follows from equalities (36) and (37) and from Lemma 3 that the normalizability of $\varphi(x)$ at $+\infty$ is equivalent to the normalizability of $\psi(x)$ at $+\infty$. Lemma 10 is proved.

4. Theorems on complete reducibility of intertwining operators

Theorem 2 (on reducibility of “dressed” nonminimizable intertwining operators).
Assume that the following conditions are satisfied:

- (1) $h^+ = -\partial^2 + V_1(x)$, $V_1(x) \in K$;
- (2) $h^- = -\partial^2 + V_2(x)$, where the potential $V_2(x)$ is real-valued and belongs to $C_{\mathbb{R}}$;
- (3) h^+ and h^- are intertwined by a nonminimizable differential operator q_N^- of N th order with coefficients from $C_{\mathbb{R}}^2$, so that

$$q_N^- h^+ = h^- q_N^-; \quad (38)$$

- (4) the algebraic multiplicity of λ_i , the i th eigenvalue of the matrix \mathbf{S} for the operator q_N^- , is equal to k_i , $i = 1, \dots, n$, so that $k_1 + \dots + k_n = N$; all of the numbers λ_i are real and satisfy the inequalities

$$0 \geq \lambda_1 > \lambda_2 > \dots > \lambda_n; \quad (39)$$

- (5) Λ is the spectrum of the matrix \mathbf{S} of the operator q_N^- ;
- (6) $E_{i\pm}$, $i = 0, 1, 2, \dots$ is the energy of the i th (from below) bound state of h^\pm , N^\pm is the number of bound states of h^\pm whose energies are included into Λ , and N_\pm is the number of bound states of h^\pm with energies not exceeding λ_1 ;
- (7)

$$P_\pm(E) = \prod_{E_{i\pm} < \lambda_1, E_{i\pm} \notin \Lambda} (E_{i\pm} - E). \quad (40)$$

Then:

- (1) $V_2(x) \in K$; coefficients of q_N^- belong to $C_{\mathbb{R}}^\infty$ and are real-valued; $q_N^+ = (q_N^-)^t$ has real-valued coefficients from $C_{\mathbb{R}}^\infty$ and intertwines h^+ and h^- , so that

$$h^+ q_N^+ = q_N^+ h^-; \quad (41)$$

- (2) $P_+(E) \equiv P_-(E)$; the degree of $P_\pm(E)$ is equal to $N_+ - N^+ = N_- - N^-$;
- (3) the operator $q_N^\mp P_\pm(h^\pm)$ intertwines h^+ and h^- and can be represented as the product of $N + N_+ + N_- - N^+ - N^-$ intertwining operators of first order with real-valued coefficients from $C_{\mathbb{R}}^\infty$, so that:
 - (a) potentials of all the intermediate Hamiltonians belong to K ;
 - (b) the eigenvalue of the matrix \mathbf{S} of the l th operator (from the right) in the factorization under consideration is equal to $E_{l-1,\pm}$, $l = 1, \dots, N_\pm$, and an element of the kernel of this operator is normalizable at both infinities;
 - (c) the eigenvalue of the matrix \mathbf{S} of the l th operator (from the left) in the factorization under consideration is equal to $E_{l-1,\mp}$, $l = 1, \dots, N_\mp$, and an element of the kernel of this operator is nonnormalizable at both infinities;

- (d) *the set of eigenvalues of the matrices \mathbf{S} for operators from the $(N_{\pm} + 1)$ th to the $(N_{\pm} + N - N^+ - N^-)$ th one (from the right) in the factorization under consideration coincides with³ $\Lambda \setminus (\{E_{i+}\} \cup \{E_{i-}\})$. In addition, the eigenvalue of the matrix \mathbf{S} for an operator of this group does not decrease as the number of the operator increases (from the right to left); a basis element of the kernel of any operator of this group is normalizable at one of the infinities only.*

Proof. The first statement of the theorem with the exception of reality of q_N^{\pm} coefficients follows from Lemma 1. The fact, that coefficients of q_N^{\pm} are real-valued, will be proved below.

To prove the second statement, it is obviously sufficient to show that if $E_{i\pm} < \lambda_1$ and $E_{i\pm} \notin \Lambda$, then there exist an $E_{j\mp}$ such that $E_{j\mp} = E_{i\pm}$. The latter fact follows from Lemmas 3 and 6. Thus, the second statement is proved.

Intertwining of h^+ and h^- by the operators $q_N^{\mp} P_{\pm}(h^{\pm})$ is evident.

By the definition of P_{\pm} and Lemma 6, the kernel $\ker(q_N^{\mp} P_{\pm}(h^{\pm}))$ contains wave functions of N_{\pm} lower bound states of h^{\pm} . Moreover, in view of the nonminimizability of q_N^{\mp} , the canonical basis in $\ker(q_N^{\mp} P_{\pm}(h^{\pm}))$ can be chosen to contain all these wave functions. Using the standard procedure described in Lemma 1 of [25], one can separate successively from the right $q_N^{\mp} P_{\pm}(h^{\pm})$ intertwining operators of first orders whose kernels bases consist of ground state wave functions of h^{\pm} or of the corresponding intermediate Hamiltonians. In addition, it is easy to verify that coefficients of separated intertwining operators are real-valued and infinitely smooth, and that potentials of intermediate Hamiltonians belong to K by induction with the help of the reasoning from the proof of Lemma 8. Thus, statement 3(b) is proved, and the ground state of the last intermediate Hamiltonian h_0^{\pm} is situated above λ_1 .

Let k_{\mp} be the remainder of $q_N^{\mp} P_{\pm}(h^{\pm})$ after realization of all the above-mentioned separations. This operator intertwines h_0^{\pm} and h^{\mp} , so that $k_{\mp} h_0^{\pm} = h^{\mp} k_{\mp}$ and $h_0^{\pm} k_{\mp}^t = k_{\mp}^t h^{\mp}$. Thus, by Lemma 3 and due to the absence of energy levels of h_0^{\mp} that are not situated above λ_1 , wave functions of N_{\mp} lower bound states of h^{\mp} belong to $\ker k_{\mp}^t$. On the other hand, the nonminimizability of q_N^{\mp} , Theorem 2 of [21], and the rule of transformation of the Jordan form of the matrix \mathbf{S} of an intertwining operator under separation from it an intertwining operator of first order (see Lemma 1 of [25]) imply that the operators k_{\mp} and thereby k_{\mp}^t are nonminimizable. Hence, wave functions of N_{\mp} lower bound states of h^{\mp} belong to a canonical basis in $\ker k_{\mp}^t$. Using the same separation procedure as above and taking into account that a product of elements of bases in $\ker(\partial - \chi(x))$ and $\ker(\partial - \chi(x))^t$ is constant, we establish statement 3(c).

Let us denote by r_{\mp} the remainder of k_{\mp} after separation of the operators mentioned in statement 3(c). This operator is nonminimizable again (by the above-mentioned rule of Jordan form transformation and Theorem 2 of [21]). Statements 3(d) and 3(a) follow from the nonminimizability of r_{\mp} , from Lemma 8 and from the fact that, by construction, r_{\mp} intertwine the Hamiltonians whose ground states are situated above λ_1 .

Coefficients of q_N^{\mp} are real-valued since coefficients of all operators contained in the obtained factorizations of $q_N^{\mp} P_{\pm}(h^{\pm})$ are real-valued, as well as coefficients of $P_{\pm}(h^{\pm})$. Theorem 2 is proved.

³In this formula, one has to take into account multiplicities of eigenvalues as follows: if λ is contained in Λ with algebraic multiplicity K_1 , in $\{E_{i+}\}$ with multiplicity K_2 , and in $\{E_{i-}\}$ with multiplicity K_3 (obviously, K_2 and K_3 can take values 0 and 1 only), then the value λ is contained in $\Lambda \setminus (\{E_{i+}\} \cup \{E_{i-}\})$ with multiplicity $K_1 - K_2 - K_3$ if $K_1 > K_2 + K_3$ or is not contained if $K_1 \leq K_2 + K_3$.

Theorem 3 (on complete reducibility of nonminimizable intertwining operators).
Assume that the following conditions are satisfied:

- (1) $h^+ = -\partial^2 + V_1(x)$, $V_1(x) \in K$;
- (2) $h^- = -\partial^2 + V_2(x)$, where the potential $V_2(x)$ is real-valued and belongs to $C_{\mathbb{R}}$;
- (3) h^+ and h^- are intertwined by a nonminimizable differential operator q_N^- of N th order with real-valued coefficients from $C_{\mathbb{R}}^2$, so that

$$q_N^- h^+ = h^- q_N^-; \quad (42)$$

- (4) the algebraic multiplicity of λ_i , the i th eigenvalue of the matrix \mathbf{S} for the operator q_N^- , is equal to k_i , $i = 1, \dots, n$, so that $k_1 + \dots + k_n = N$; the set of values λ_i contains M real values and L pairs of mutually complex conjugate ones, so that $M + 2L = n$; the indices $i = 1, \dots, M$ correspond to real λ_i , and $\lambda_i > \lambda_{i+1}$, $i = 1, \dots, M - 1$;
- (5) if λ_1 is real, then $\lambda_1 \leq 0$;
- (6) $E_{i\pm}$, $i = 0, 1, 2, \dots$, is the energy of the i th (from below) bound state of h^\pm ; $K_\pm = \max\{i : \lambda_i > E_{0\pm}\}$ if $\lambda_1 > E_{0\pm}$, and $K_\pm = 0$ if either $\lambda_1 \leq E_{0\pm}$ or $\text{Im } \lambda_1 \neq 0$.

Then:

- (1) $V_2(x) \in K$; coefficients of q_N^- belong to $C_{\mathbb{R}}^\infty$; $q_N^+ = (q_N^-)^t$ has real-valued coefficients from $C_{\mathbb{R}}^\infty$ and intertwines h^+ and h^- , so that

$$h^+ q_N^+ = q_N^+ h^-; \quad (43)$$

- (2) q_N^\mp can be represented as a product of really irreducible intertwining operators of first and second orders with real-valued coefficients from $C_{\mathbb{R}}^\infty$, so that:

- (a) potentials of all the intermediate Hamiltonians belong to K ;
- (b) the first

$$J_1 = \sum_{i=M+1}^{M+L} k_i \quad (44)$$

operators from the right in the factorization of q_N^\mp under consideration have order two and are really irreducible operators of the I type; in addition, one can realize that the related to these operators pairs of mutually complex conjugate eigenvalues of the matrix \mathbf{S} for the operator q_N^- are ordered arbitrarily;

- (c) the second (from the right) group of operators in the factorization under consideration consists of

$$J_{2\mp} = N - 2J_1 - 2J_{3\mp}, \quad (45)$$

operators of first order, where

$$J_{3\mp} = \left[\frac{1}{2} \sum_{i=1}^{K_\mp} k_i \right], \quad (46)$$

and

- (i) if $\sum_{i=1}^{K_\mp} k_i$ is even, then the eigenvalue of the matrix \mathbf{S} for the operator q_N^- which corresponds to the l th (from the right) of these operators does not exceed the eigenvalue related to the $(l + 1)$ th operator, $l = 1, \dots, J_{2\mp} - 1$;

- (ii) if $\sum_{i=1}^{K_{\mp}} k_i$ is odd, then the eigenvalue of the matrix \mathbf{S} for the operator q_N^- which corresponds to the l th (from the right) of these operators does not exceed the eigenvalue related to the $(l+1)$ th operator, $l = 1, \dots, J_{2\mp} - 2$; $\lambda_{K_{\mp}}$ is the eigenvalue that corresponds to the $(J_{2\mp} - 1)$ th operator and $\lambda_{K_{\mp}+1}$ is the eigenvalue that corresponds to the $J_{2\mp}$ th operator; in this case, the latter eigenvalue is equal to $E_{0\mp}$;
- (d) the third (from the right) and the last group of operators in the factorization under consideration consists of $J_{3\mp}$ really irreducible operators of the II and III types, and the largest of eigenvalues of the matrix \mathbf{S} for the operator q_N^- which corresponds to the l th of these operators (from the right) does not exceed the smallest eigenvalue of the matrix \mathbf{S} for the operator q_N^- which corresponds to the $(l+1)$ th of these operators, $l = 1, \dots, J_{3\mp} - 1$.

Remark 1. If $E_{0\mp}$ is not an eigenvalue of the matrix \mathbf{S} for the operator q_N^- , then $\sum_{i=1}^{K_{\mp}} k_i$ is even since otherwise the eigenvalue of $q_N^{\mp} q_N^{\pm} \equiv \prod_{i=1}^n (h^{\mp} - \lambda_i)^{k_i}$ at the ground state wave function of h^{\mp} is negative.

Proof. Let us restrict ourselves by a proof for the case of q_N^- only (a proof for the case of q_N^+ is analogous). The first statement follows from Lemma 1. Statement 2(b) follows from Lemma 7. Statement 2(c) in the part that corresponds to intertwining operators for which the eigenvalues of the matrices \mathbf{S} are situated below E_{0-} follows from Lemma 8. It also follows from Lemmas 7 and 8 that corresponding part of intermediate Hamiltonians belongs to K . Thus, the proof is reduced to the case where $L = 0$ and $\lambda_M = \lambda_n \geq E_{0-}$, which is assumed below.

Let us first describe the main idea of the proof. The idea is as follows. We apply Theorem 2 to factorize the operator $q_N^- P_+(h^+)$ into three groups of intertwining operators of first order. Then we successively permute any operator from the right-hand group (by Lemmas 9 and 10) with the operators of the middle group (certainly, such an operator is changed by any permutation, but its matrix \mathbf{S} is preserved) until this operator either takes its proper position in the middle group (if the eigenvalue of its matrix \mathbf{S} belongs to the spectrum of the matrix \mathbf{S} of the operator q_N^-) or pass the middle group entirely. In parallel, one must permute operators from the left-hand group (such that eigenvalues of their matrices \mathbf{S} belong to the spectrum of the matrix \mathbf{S} of the operator q_N^-) with operators of the middle group as long as they get their proper positions. In this connection, operators of the right-hand group that pass the middle group entirely will form, under contact with operators of the left-hand group with the same matrices \mathbf{S} , differences of eigenvalues and Hamiltonians which provide the possibility to minimize $q_N^- P_+(h^+)$ to q_N^- , and thus, to get the required factorization of q_N^- as a result.

Now we present details. We consider successively (from top to bottom) all the energy levels of the super-Hamiltonian H that are not situated above λ_1 . We start from the case of the upper of these levels, E_{\max} . If E_{\max} coincides with one of eigenvalues of the matrix \mathbf{S} of the operator q_N^- (so that $E_{\max} = \lambda_i$), then we proceed as follows.

- (a) If E_{\max} belongs to the spectrum of h^+ , then we permute the corresponding to E_{\max} operator from the right-hand group of the factorization given by Theorem 2 (obviously, this operator is the most left in the right-hand group) with operators of the

middle group from right to left with the help of Lemma 9 until the permutation with the most left of the operators that correspond to λ_{i+1} .

- (b) If E_{\max} does not belong to the spectrum of h^+ , then cofactors of the right-hand group contain no cofactor corresponding to E_{\max} , and we do not make any permutations from right to left with cofactors of the middle group.
- (c) If E_{\max} belongs to the spectrum of h^- , then we permute the corresponding to E_{\max} operator from the left-hand group (obviously, this operator is the most right in the left-hand group) with operators of the middle group from left to right with the help of Lemma 10 either until the permutation with the most right of the operators corresponding to λ_{i-1} (if $k_1 + \dots + k_{i-1}$ is even) or until the permutation after which the right-hand neighbour of the moved operator is the most right of the operators corresponding to λ_{i-1} (if $k_1 + \dots + k_{i-1}$ is odd). Let us note that in the case under consideration, the following happens. If two first order operators are to the right of the moved operator from the left-hand group before a permutation and are to the left from the moved operator after the permutation, then these two operators form the really irreducible second order operator of the II or III type. This is explained by the fact that both eigenvalues of the matrix \mathbf{S} of this operator are situated after the permutation above the energy of the ground state (generated by the moved operator from the left-hand group since an element of its kernel is nonnormalizable at both infinities by Theorem 2 and Lemma 10); thus, both elements of a canonical basis in the kernel of the considered operator, which are formal eigenfunctions of the proper intermediate Hamiltonian, must have zeroes.
- (d) If E_{\max} does not belong to the spectrum of h^- , then cofactors of the left-hand group contain no cofactor corresponding to E_{\max} , and we do not make any permutations from left to right with cofactors of the middle group.

Now we consider the case where E_{\max} does not belong to the spectrum of the matrix \mathbf{S} of the operator q_N^- . In this case, there is the index i such that $\lambda_i > E_{\max} > \lambda_{i+1}$ (or $i = n = M$ and $\lambda_i > E_{\max}$). In addition, in this case, both Hamiltonians h^\pm have the level E_{\max} (see the second statement of Theorem 2); to avoid a negative eigenvalue of the supercharges anticommutator for a wave function of H for the level E_{\max} , the following condition must hold:

$$k_1 + \dots + k_i \quad \text{is even.} \quad (47)$$

In the considered case, the most left of the right-hand group cofactors corresponds to E_{\max} . We permute this cofactor with the help of Lemma 9 with cofactors of the middle group until the permutation with the most left of the cofactors corresponding to λ_{i+1} . Further permutations are accomplished with the help of Lemma 10. In this connection, the passage of the considered operator from the right-hand group through the entire middle group is possible by virtue of condition (47). Let us note that after each permutation with the help of Lemma 10, the right-hand neighbour of the moved operator is the united really irreducible operator of the II or III type and not two separate operators of first order. This is explained by the fact that after the permutation, both eigenvalues of the matrix \mathbf{S} of this neighbour are situated above the energy of the ground state (generated by the moved operator from the right-hand group since an element of its kernel is normalizable at both infinities by Theorem 2 and Lemmas 9 and 10); thus, both elements of a canonical basis in the kernel of the considered operator, which are formal eigenfunctions of the proper intermediate Hamiltonian, must have zeroes.

After passing through the middle group, the operator of the right-hand group is located near the intertwining operator of the left-hand group with the same matrix \mathbf{S} . By Theorem 2 of [21] and the rule of transformation of the Jordan form of the matrix \mathbf{S} of an intertwining operator under separation from it an intertwining operator of first order (see the proof of Lemma 1 in [25]), the product of these operators is equal to the difference of E_{\max} and the intermediate Hamiltonian. With the help of intertwining relations, this difference can be moved to the bound of the considered factorization and separated.

We proceed further in the same way by induction. As a result, we obtain the required factorization of q_N^- . Theorem 3 is proved.

Acknowledgments

The author thanks A.A. Andrianov for numerous discussions and useful remarks. This research was supported by the RFBR (project 06-01-00186-a).

References

- [1] F. Cooper, B. Freedman, — Ann. Phys. (NY) **146** (1983), 262.
- [2] A.A. Andrianov, N.V. Borisov, M.V. Ioffe, — JETP Lett. **39** (1984), 93; Phys. Lett. A **105** (1984), 19; Teor. Mat. Fiz. **61** (1984), 1078.
- [3] B. Mielnik, — J. Math. Phys. **25** (1984), 3387.
- [4] M.M. Nieto, — Phys. Lett. B **145** (1984), 208.
- [5] D.J. Fernández C., — Lett. Math. Phys. **8** (1984), 337 [physics/0006119].
- [6] A. Lahiri, P.K. Roy, B. Bagchi, — Int. J. Mod. Phys. A **5** (1990), 1383.
- [7] F. Cooper, A. Khare, U. Sukhatme, — Phys. Rept. **251** (1995), 267.
- [8] V.G. Bagrov, B.F. Samsonov, — Phys. Part. Nucl. **28** (1997), 374.
- [9] A.A. Andrianov, M.V. Ioffe, V.P. Spiridonov, — Phys. Lett. A **174** (1993), 273.
- [10] E. Schrödinger, — Proc. Roy. Irish Acad. A **47** (1941), 53 [physics/9910003].
- [11] L. Infeld, T.E. Hull, — Rev. Mod. Phys. **23** (1951), 21.
- [12] A.A. Andrianov, N.V. Borisov, M.I. Eides, M.V. Ioffe, — **61** (1984), 965; Phys. Lett. A **109** (1985), 143.
- [13] C.V. Sukumar, — J. Phys. A: Math. Gen. **18** (1985), L57; 2917.
- [14] M.M. Crum, — Quart. J. Math. (Oxford) **6** (1955), 121 [physics/9908019].
- [15] M.G. Krein, — Dokl. Akad. Nauk SSSR, **113** (1957), 970.
- [16] L.D. Faddeev, — Usp. Mat. Nauk, **14** (1959), 57 [J. Math. Phys. **4** (1963), 72].
- [17] V.B. Matveev, M. Salle, *Darboux transformations and solitons*. — Springer, Berlin (1991).
- [18] A.A. Andrianov, F. Cannata, J.-P. Dedonder, M.V. Ioffe, — Int. J. Mod. Phys. A **10** (1995), 2683.
- [19] B.F. Samsonov, — Phys. Lett. A **263** (1999), 274.
- [20] A.A. Andrianov, F. Cannata, — J. Phys. A: Math. Gen. **37** (2004), 10297.

- [21] A.A. Andrianov, A.V. Sokolov, — J. Math. Sci. **143**(1) (2007), 2707 (translated from Zapiski Nauchnykh Seminarov POMI **335** (2006), 22); arXiv: 0710.5738.
- [22] A.A. Andrianov, F. Cannata, A.V. Sokolov, — Nucl. Phys. B **773** [PM] (2007), 107; math-ph/0610024.
- [23] A.V. Sokolov, — Nucl. Phys. B **773** [PM] (2007), 137; math-ph/0610022.
- [24] D.J. Fernández C., R. Muñoz and A. Ramos, — Phys. Lett. A **308** (2003), 11; quant-ph/0212026.
- [25] A.A. Andrianov, A.V. Sokolov, — Nucl. Phys. B **660** (2003), 25; hep-th/0301062.
- [26] F.A. Berezin, M.A. Shubin, *Schrödinger Equation*. — MSU Publ., Moscow (1983).